Beck's monadicity in a 2-derivator

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Introduction

Monads can be defined in a general 2-category $\mathcal{C}$ [5] and they correspond to 2-functors from $\mathbf{Mnd} := \mathbf{B}\Delta_+$ to $\mathcal{C}$, while Eilenberg-Moore objects can be described as weighted limits [8]. By taking nerves hom-wise one gets homotopy coherent monads and ∞-categories of algebras in a quasi-categorically enriched category $\mathcal{K}$ [6].

It can be shown that $[-,\mathcal{C}]$ and $h_*[N_*-,\mathcal{K}]$ are 2-derivators. Can one talk about monads and algebras inside a general 2-derivator $\mathbb{D}$? The answer is affirmative. An homotopy coherent monad is an object of the 2-category $\mathbb{D}(\mathbf{Mnd})$ and algebras can be recovered using Kan extensions along the inclusion of **Mnd** inside the free adjunction **Adj** (see box on the right).

What about monadicity? Given an homotopy coherent adjunction $\mathbb{A} \in \mathbb{D}(\mathbf{Adj})$, there exists a canonical comparison 1-cell to the Eilenberg-Moore adjunction as a consequence of the axioms of 2-derivators. This comparison is an equivalence provided that $\mathbb{A}$ satisfies some conditions, which are analogous to those of Beck's monadicity theorem.

What is a 2-derivator?

A *derivator* [3] (also called a homotopy theory in [4]) is a 2-functor $\mathbf{Cat}^{\mathrm{op}} \to \mathbf{CAT}$ subject to axioms formalising the key features of the assignment $I \mapsto \mathrm{Ho}(\mathcal{M}^I)$ for a nice model category $\mathcal{M}$, in particular it allows the calculus of homotopy Kan extensions.

2-derivators [1] extend the theory of derivators: we are interested in the *homotopy 2-category* of an enriched model category, the base of enrichment being the category of simplicial sets with the *Joyal model structure.*

A 2-derivator is a (weak) 3-functor $\mathbb{D}\colon \mathbf{2}\text{-}\mathbf{Cat}^{\mathrm{op}} \to \mathbf{GRAY}$ satisfying some axioms, including the following which are paramount in the theory.

- For every $X \in \mathbb{D}(\mathbb{1})$ and every cospan $B \xrightarrow{f} A \xleftarrow{g} C$ there exists a *smothering functor* [7]

$$\mathbb{D}(\mathbb{1})(X, f \downarrow g) \to \mathbb{D}(\mathbb{1})(X,f) \downarrow \mathbb{D}(\mathbb{1})(X,g)$$

  enabling a characterization of commas, which specialises to weak comma objects in ∞-cosmoi.
- The 2-functor (defined using the closeness of **GRAY**) relating external and internal adjunctions

$$\mathrm{dia}_{\mathbf{Adj}}\colon \mathbb{D}(\mathbf{Adj}) \to \mathbb{D}(\mathbb{1})^{\mathbf{Adj}}$$

  is a *smothering 2-functor*, namely surjective on objects and locally smothering.
- For every 2-functor $u\colon \mathcal{C} \to \mathcal{D}$ there exist biadjunctions

$$\mathrm{Lan}_u \dashv_b u^* := \mathbb{D}(u) \dashv_b \mathrm{Ran}_u$$

  and they can be computed in a pointwise way by using collages.

Why study 2-derivators?

They allow a unified treatment of higher categories, providing a context in which to talk about ∞-categories indipendently of the precise $(\infty,2)$-categorical model used to present them. Therefore, they are well suited to capture more models of ∞-categories as well as to systematically describe enriched, internal and fibered ∞-categories in a synthetic way.

The collage construction and its applications

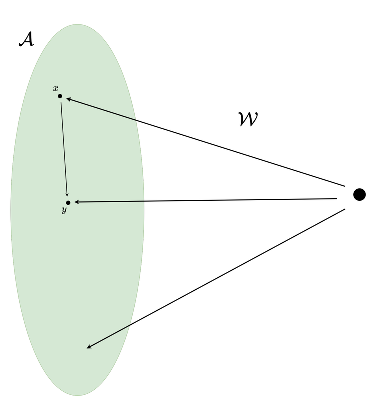


Figure 1: The 2-category coll$\mathcal{W}$.

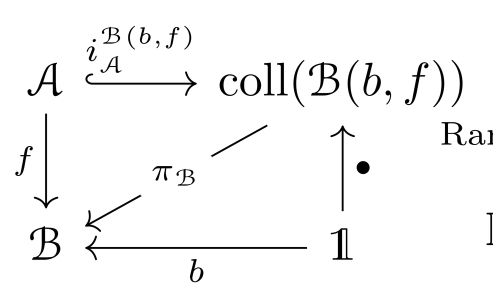


Figure 2: Diagram for pointwise Kan extensions of 2-derivators.

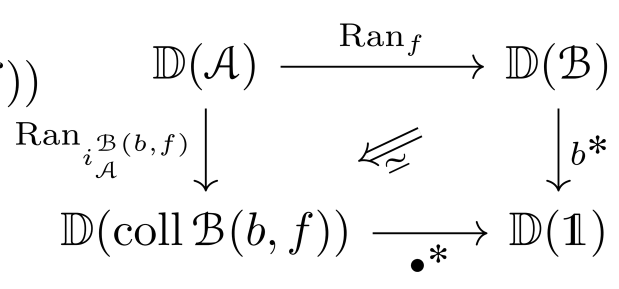


Figure 3: 2-cell expressing pointwise Kan extensions.

The collage coll$\mathcal{W}$ of the weight $\mathcal{W}\colon \mathcal{A} \to \mathbf{Cat}$ is the 2-category with objects $\mathrm{Ob}(\mathcal{A}) \sqcup \{\bullet\}$, hom-categories $\mathcal{A}(x,y)$ between $x, y \in \mathrm{Ob}(\mathcal{A})$, $\mathcal{W}x$ from $\bullet$ to $x \in \mathcal{A}$, $\mathbb{1}$ on $\bullet$ and the empty category elsewhere (see Figure 1). Composition comes from the one in $\mathcal{A}$ and the functoriality of $\mathcal{W}$.

This construction allows to capture weighted limits in a 2-derivator, generalising a classical result [7] ($i_{\mathcal{A}}^{\mathcal{W}}$ is the inclusion of $\mathcal{A}$ in the collage):

$$\lim^{\mathcal{W}} := \mathbb{D}(\mathcal{A}) \xrightarrow{\mathrm{Ran}_{i_{\mathcal{A}}^{\mathcal{W}}}} \mathbb{D}(\mathrm{coll}\mathcal{W}) \xrightarrow{\bullet^*} \mathbb{D}(\mathbb{1}).$$

Collages are also a powerful tool to interpret pointwise Kan extensions in an enriched setting, where weighted limits cannot in general be written using only conical limits. The diagram in **2-Cat** of Figure 2 is indeed sent by $\mathbb{D}$ to a diagram in **GRAY** with specified biadjoints. Using the calculus of mates in **GRAY** (an immediate consequence of the pasting theorem in [2]) one can define Kan extensions in a pointwise fashion by asking for a certain 2-cell (see Figure 3)

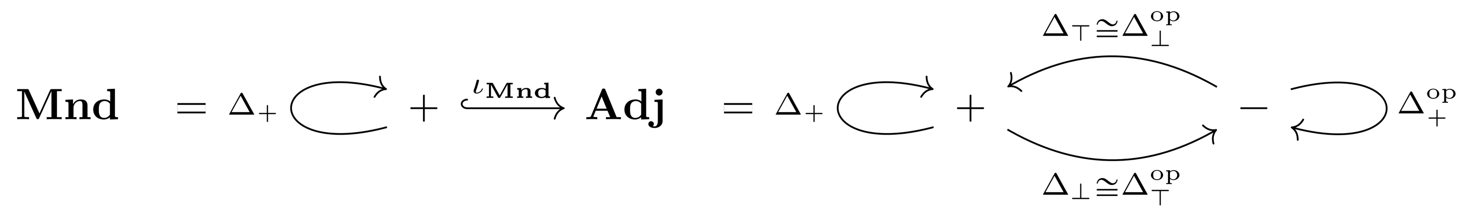
$$b^* \mathrm{Ran}_f \Rightarrow \bullet^* \mathrm{Ran}_{i_{\mathcal{A}}^{\mathcal{B}(b,f)}}$$

to be an equivalence.

Eilenberg-Moore objects via collages

Just like a monad in a 2-category can be seen as a 2-functor from **Mnd**; an adjunction is also described as a 2-functor from the free adjunction **Adj**, which contains **Mnd** as a full sub-2-category.

Generalising, for $\mathbb{A} \in \mathbb{D}(\mathbf{Adj})$ one can recover the object of algebras for its underlying monad as the right Kan extension along the inclusion $\iota_{\mathbf{Mnd}}$ evaluated at $-$, a.k.a. $-^* \mathrm{Ran}_{\iota_{\mathbf{Mnd}}} \iota^*_{\mathbf{Mnd}} \mathbb{A}$.



Since Kan extensions are weighted limits, one can also obtain algebras by means of the collage construction. In this case the collage (shown below) consists only of two objects. Notice that the "$-$" in the first variable is one of the objects of **Adj** and not a placeholder!

Internalisation of monads

Adjunctions can be characterised by a universal property that is inferable from external diagrams and hence can be internalised [6]. This motivates the axiom about $\mathrm{dia}_{\mathbf{Adj}}$. On the contrary, monads don't have a universal property but they are only defined algebraically. In the words of 2-derivators, $\mathrm{dia}_{\mathbf{Mnd}}$ is not smothering.

However, given an adjunction in $\mathbb{D}(\mathbb{1})$ one can lift it to an homotopy coherent adjunction using $\mathrm{dia}_{\mathbf{Adj}}$ and then apply $\iota^*_{\mathbf{Mnd}}$ to get an homotopy coherent monad:

$$\mathbb{D}(\mathbf{Mnd}) \underset{\mathrm{Ran}_{\iota_{\mathbf{Mnd}}}}{\overset{\iota^*_{\mathbf{Mnd}}}{\underset{\perp}{\rightleftarrows}}} \mathbb{D}(\mathbf{Adj}) \xrightarrow{\mathrm{dia}_{\mathbf{Adj}}} \mathbb{D}(\mathbb{1})^{\mathbf{Adj}}$$

Beck's monadicity: the conditions

The conditions of Beck's monadicity for an $\mathbb{A} \in \mathbb{D}(\mathbf{Adj})$ can be expressed in the 2-category $\mathbb{D}(\mathbb{1})$.

- Objects of $\mathbb{D}(\mathbb{1})$ admit cotensors with small categories, computed using collages.
- The underlying adjunction $f \dashv u\colon B := +^*\mathbb{A} \rightleftarrows A := -^*\mathbb{A}$ is the image of the distinguished adjunction [7] in **Adj**.
- The object of u-split simplicial objects [7] can be obtained as the pullback of the cospan

$$A^{\Delta^{\mathrm{op}}} \xrightarrow{u^{\Delta^{\mathrm{op}}}} B^{\Delta^{\mathrm{op}}} \xleftarrow{\mathrm{res}} B^{\Delta_\top}$$

- The conditions on A admitting colimits of u-split simplicial objects, u preserving them and being conservative can be interpreted in $\mathbb{D}(\mathbb{1})$ in the standard 2-categorical way.

Beck's monadicity: sketch of the proof

- Use the unit of the biadjunction $\iota^*_{\mathbf{Mnd}} \dashv_b \mathrm{Ran}_{\iota_{\mathbf{Mnd}}}$ to find the comparison 1-cell.
- Externalise diagrams using the 2-functor $\mathrm{dia}_{\mathbf{Adj}}\colon \mathbb{D}(\mathbf{Adj}) \to \mathbb{D}(\mathbb{1})^{\mathbf{Adj}}$ to prove that the image of the comparison is an equivalence, then use its smothering property to conclude.
- Kan extensions along fully faithful 2-functors are true extensions [1]: the component at $+$ of the externalised unit is always an equivalence.
- Adapt the argument in [7] to prove that the component at $-$ is an equivalence when the conditions of Beck's monadicity are satisfied.

Future work

- Descent in a 2-derivator.
- ∞-operads as analytic monads in a 2-derivatorial setting.
- Synthetic enriched higher category theory.

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